

# MINIMAL COLORING NUMBER FOR $\mathbb{Z}$ -COLORABLE LINKS

KAZUHIRO ICHIHARA AND ERI MATSUDO

**ABSTRACT.** For a link with zero determinants, a  $\mathbb{Z}$ -coloring is defined as a generalization of Fox coloring. We call a link having a diagram which admits a non-trivial  $\mathbb{Z}$ -coloring a  *$\mathbb{Z}$ -colorable link*. The *minimal coloring number* of a  $\mathbb{Z}$ -colorable link is the minimal number of colors for non-trivial  $\mathbb{Z}$ -colorings on diagrams of the link. We give sufficient conditions for non-splittable  $\mathbb{Z}$ -colorable links to have the least minimal coloring number.

## 1. INTRODUCTION

In [2], Fox introduced one of the most well-known invariants for knots and links, now called *the Fox  $n$ -coloring*, or simply  $n$ -coloring for  $n \geq 2$ .

For a link  $L$ , if the determinant of  $L$  is 0, then it is shown that  $L$  admits no non-trivial  $n$ -coloring for any  $n \geq 2$ . Please refer to [5] for the definition of the determinant of a link for example. In that case,  $L$  admits a  $\mathbb{Z}$ -coloring defined as follows.

**Definition 1.1.** Let  $L$  be a link and  $D$  a regular diagram of  $L$ . We consider a map  $\gamma : \{\text{arc of } D\} \rightarrow \mathbb{Z}$ . If  $\gamma$  satisfies the condition  $2\gamma(a) = \gamma(b) + \gamma(c)$  at each crossing of  $D$  with the over arc  $a$  and the under arcs  $b$  and  $c$ , then  $\gamma$  is called a  *$\mathbb{Z}$ -coloring* on  $D$ . A  $\mathbb{Z}$ -coloring which assigns the same color to all the arcs of the diagram is called the *trivial  $\mathbb{Z}$ -coloring*. A link is called  *$\mathbb{Z}$ -colorable* if it has a diagram admitting a non-trivial  $\mathbb{Z}$ -coloring.

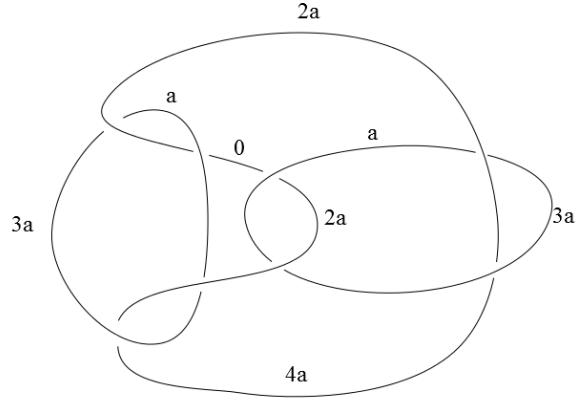
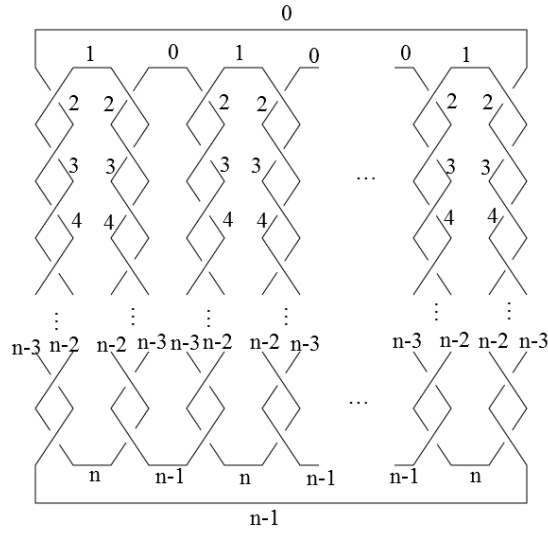
The links illustrated in Figure 1 and 2 are examples those are  $\mathbb{Z}$ -colorable. Throughout this paper, we adopt the names of links as those given in [1].

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FIGURE 1.  $L8n6$  ( $a \geq 1$ )FIGURE 2. Pretzel link  $P(n, -n, \dots, n, -n)$  for  $n \geq 1$ 

In [4], Harary and Kauffman defined the minimal coloring number for links with Fox colorings. Here we define the minimal coloring number for  $\mathbb{Z}$ -colorable links as a generalization.

**Definition 1.2.** Let us consider the cardinality of the image of a non-trivial  $\mathbb{Z}$ -coloring on a diagram of a  $\mathbb{Z}$ -colorable link  $L$ . We call the minimum of such cardinalities among all non-trivial  $\mathbb{Z}$ -colorings on all diagrams of  $L$  the *minimal coloring number* of  $L$ , and denote it by  $\text{mincol}_{\mathbb{Z}}(L)$ .

In this paper, we give sufficient conditions for non-splittable  $\mathbb{Z}$ -colorable links to have the least minimal coloring number. First, after some preliminaries in Section 2, for a non-splittable  $\mathbb{Z}$ -colorable link  $L$ , we show that  $\text{mincol}_{\mathbb{Z}}(L) \geq 4$  in Section 3. Next, we introduce a *simple*  $\mathbb{Z}$ -coloring in Section 4, and show that if a link  $L$  admits a simple  $\mathbb{Z}$ -coloring, then  $\text{mincol}_{\mathbb{Z}}(L) = 4$ . Also, in the case that a link  $L$  admits a  $\mathbb{Z}$ -coloring by five colors, then we have  $\text{mincol}_{\mathbb{Z}}(L) = 4$ , which is shown in Section 5.

## 2. PRELIMINARIES

Throughout this paper, for a  $\mathbb{Z}$ -coloring  $\gamma$ , we represent that *the colors at a crossing are*  $\{a|b|c\}$  if the over arc is colored by  $b$  and the under arcs are colored by  $a$  and  $c$  by  $\gamma$  at the crossing.

In this section, we prepare some basic properties of  $\mathbb{Z}$ -colorings.

**Lemma 2.1.** *For any  $\mathbb{Z}$ -colorable link, there exists a  $\mathbb{Z}$ -coloring  $\gamma$  such that  $\text{Im}(\gamma) = \{0, a_1, a_2, \dots, a_n\}$  with  $a_i > 0$  ( $i = 1, 2, \dots, n$ ) for some positive integer  $n$ .*

*Proof.* We assume that there exists a  $\mathbb{Z}$ -coloring  $\gamma'$  such that  $\text{Im}(\gamma') = \{\alpha, b_1, b_2, \dots, b_n\}$  with  $b_i > \alpha$  ( $i = 1, 2, \dots, n$ ) for some positive integer  $n$ . We consider a map  $\gamma : \{\text{arc of the diagram}\} \rightarrow \mathbb{Z}$  with  $\text{Im}(\gamma) = \{0, a_1, a_2, \dots, a_n\}$  obtained by setting  $a_i = b_i - \alpha$ . Then at a crossing on a diagram of the link, the colors  $\{p|q|r\}$  are transformed to  $\{p-\alpha|q-\alpha|r-\alpha\}$ . We see  $(p-\alpha) + (r-\alpha) = 2(q-\alpha)$  from  $p+r = 2q$ . Therefore the map  $\gamma$  is a  $\mathbb{Z}$ -coloring such that  $\text{Im}(\gamma) = \{0, a_1, a_2, \dots, a_n\}$  with  $a_i > 0$  ( $i = 1, 2, \dots, n$ ).  $\square$

**Lemma 2.2.** *For a  $\mathbb{Z}$ -coloring  $\gamma$  with  $0 = \min \text{Im}(\gamma)$ , if an over arc at a crossing is colored by 0, then the under arcs at the crossing are colored by 0.*

*Proof.* Suppose that a crossing has colors  $\{a|b|c\}$  with  $a \neq b \neq c$  for a  $\mathbb{Z}$ -coloring  $\gamma$  with  $0 = \min \text{Im}(\gamma)$ . Then we obtain  $a > b > c$  or  $c > b > a$ . If we suppose that  $b = 0$ , then it gives  $0 > c$  or  $0 > a$ , contradicting to  $0 = \min \text{Im}(\gamma)$ .  $\square$

**Lemma 2.3.** *For a  $\mathbb{Z}$ -coloring  $\gamma$  with  $M = \max \text{Im}(\gamma)$ , if an over arc at a crossing is colored by  $M$ , then the under arcs at the crossing are colored by  $M$ .*

*Proof.* Suppose that a crossing has colors  $\{a|b|c\}$  with  $a \neq b \neq c$  for a  $\mathbb{Z}$ -coloring  $\gamma$  with  $M = \max \text{Im}(\gamma)$ . Then we obtain  $a > b > c$  or  $c > b > a$ . If we suppose that  $b = M$ , then it gives  $M < c$  or  $M < a$ , contradicting to  $M = \max \text{Im}(\gamma)$ .  $\square$

3.  $\mathbb{Z}$ -COLORINGS BY FOUR COLORS

In this section, we consider the case that a link can be colored by four colors. First, we can see that any splittable link  $L$  is  $\mathbb{Z}$ -colorable and  $\text{mincol}_{\mathbb{Z}}(L) = 2$ . On the other hand, the next holds for non-splittable links.

**Theorem 3.1.** *Let  $L$  be a non-splittable  $\mathbb{Z}$ -colorable link. Then  $\text{mincol}_{\mathbb{Z}}(L) \geq 4$ .*

*Proof.* Let  $\gamma$  be a non-trivial  $\mathbb{Z}$ -coloring for a  $\mathbb{Z}$ -colorable link  $L$ . We will show that if  $\text{mincol}_{\mathbb{Z}}(L) \leq 3$ , then  $L$  is splittable.

Since  $\gamma$  is non-trivial, the cardinality of  $\text{Im}(\gamma)$  is greater than 1.

In the case that the cardinality of  $\text{Im}(\gamma)$  is 2, we can assume that  $\text{Im}(\gamma) = \{0, a\}$  by Lemma 2.1. By Lemma 2.3 a diagram of  $L$  has no crossings with over arcs labeled by  $a$  other than  $\{a|a|a\}$ . And it also has no crossings with over arcs labeled by 0 other than  $\{0|0|0\}$ . Therefore the diagram is splittable, and we see that the link is splittable.

In the case that the cardinality of  $\text{Im}(\gamma)$  is 3, we can assume  $\text{Im}(\gamma) = \{0, a, b\}$  by Lemma 2.1. Let  $b$  be the maximum of  $\text{Im}(\gamma)$ . By Lemma 2.3, a diagram of the link has no crossings with over arcs labeled by  $b$  other than  $\{b|b|b\}$ . If the diagram has no crossing colored by  $\{0|a|b\}$ , it has only crossings colored by  $\{0|0|0\}$ ,  $\{a|a|a\}$  or  $\{b|b|b\}$ . Then we obtain the link is splittable. If the diagram has a vertex colored by  $\{0|a|b\}$ , we see  $b = 2a$ . From Lemma 2.2 and 2.3, the diagram has no crossings labeled by  $\{a|*|*\}$  with the exception of crossings labeled by  $\{a|a|a\}$ . Then we see that the link is splittable.  $\square$

If a link is  $\mathbb{Z}$ -colorable with four colors, we can show the following.

**Theorem 3.2.** *If  $\text{mincol}_{\mathbb{Z}}(L) = 4$  for a  $\mathbb{Z}$ -colorable link  $L$ , then there exists a diagram  $D$  of  $L$  and a  $\mathbb{Z}$ -coloring  $\gamma$  on  $D$  such that  $\text{Im}(\gamma) = \{0, 1, 2, 3\}$ .*

*Proof.* Suppose that there exists a  $\mathbb{Z}$ -coloring  $\gamma'$  with four colors. From Lemma 2.1, we can suppose  $\text{Im}(\gamma') = \{0, a, b, c\}$  with  $a > 0, b > 0$  and  $c > 0$ . We assume that  $c$  is the maximum of  $\text{Im}(\gamma')$ . Since  $L$  is a non-splittable link, we assume  $0 < a < b < c$ . From Lemma 2.2 and Lemma 2.3,  $D$  has a crossing colored by  $\{0|a|b\}$  or a crossing colored by  $\{0|b|c\}$ . If  $D$  has both of the crossings, by the definition of  $\mathbb{Z}$ -coloring, we obtain  $a - 0 = b - a$  and  $b - 0 = c - b$ . We see  $b = 2a$  and  $c = 2b$ , and then we obtain  $\text{Im}(\gamma') = \{0, a, 2a, 4a\}$ . Then the diagram has no crossings with colors  $\{a|*|*\}$  other than  $\{a|a|a\}$ . This gives a contradiction to that  $L$  is a non-splittable link. If  $D$  has only either of the crossings colored by  $\{0|a|b\}$  or  $\{0|b|c\}$ , then  $D$  has only 3 colors, contradicting

to Theorem 3.1. From the above,  $D$  has either of the crossings and another crossing colored by  $\{a|b|c\}$ . If  $D$  has a crossing colored by  $\{0|b|c\}$  and a crossing colored by  $\{a|b|c\}$ , we obtain  $b - 0 = c - b$  and  $b - a = c - b$ . We see  $a = 0$ , and this is contradictory to the assumption. If  $D$  has a crossing colored by  $\{0|a|b\}$  and a crossing colored by  $\{a|b|c\}$ , we obtain  $a - 0 = b - a$  and  $b - a = c - b$ . We see  $b = 2a$  and  $c = 3a$ . Therefore, we obtain  $Im(\gamma') = \{0, a, 2a, 3a\}$ . By dividing by  $a$ , we have a  $\mathbb{Z}$ -coloring with colors  $\{0, 1, 2, 3\}$  as desired.  $\square$

Among links of crossing numbers at most 9, there are 5 links with zero determinant. For the  $\mathbb{Z}$ -colorable links, the colorings on the diagrams in [1] are quite distinctive. See figures in Section 6. In the next section, we consider such distinctive colorings.

#### 4. REDUCTION OF COLORS

In this section, we focus on the “simplest”  $\mathbb{Z}$ -coloring found for the links with at most 9 crossings. Based on such examples, we introduce the following notion.

**Definition 4.1.** Let  $L$  be a non-trivial  $\mathbb{Z}$ -colorable link, and  $\gamma$  a  $\mathbb{Z}$ -coloring on a diagram  $D$  of  $L$ . Suppose that there exists a natural number  $d$  such that, at all the crossings in  $D$ , the differences between the colors of the over arcs and the under arcs are  $d$  or 0. Then we call  $\gamma$  a *simple*  $\mathbb{Z}$ -coloring.

For example, a pretzel knot  $P(n, -n, n, -n, \dots, n, -n)$  with integer  $n$  admits a simple  $\mathbb{Z}$ -coloring. See Figure 2.

The next is our first main result in this paper.

**Theorem 4.2.** *Let  $L$  be a non-splittable  $\mathbb{Z}$ -colorable link. If there exists a simple  $\mathbb{Z}$ -coloring on a diagram of  $L$ , then  $mincol_{\mathbb{Z}}(L) = 4$ .*

Throughout the rest of the paper, we depict a crossing colored by  $\{a|a|a\}$  with an integer  $a$  as shown in Figure 3.

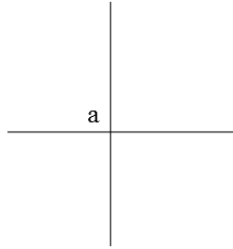


FIGURE 3.

*Proof of Theorem 4.2.* Let  $\gamma$  be a simple  $\mathbb{Z}$ -coloring on a diagram of a non-splittable  $\mathbb{Z}$ -colorable link  $L$ . Then there exists a natural number  $d$  such that, at all the crossing in  $D$ , the differences between the colors of the over arcs and the under arcs are  $d$  or  $0$ . From Lemma 2.2, we regard that  $0$  is the minimum of  $Im(\gamma)$ . For the maximum  $M$  of  $Im(\gamma)$ , by Lemma 2.3, the diagram has only crossings colored by  $\{M|M|M\}$  or  $\{M|M-d|M-2d\}$ . First we delete a crossing colored by  $\{M|M|M\}$  as shown in Figure 4.

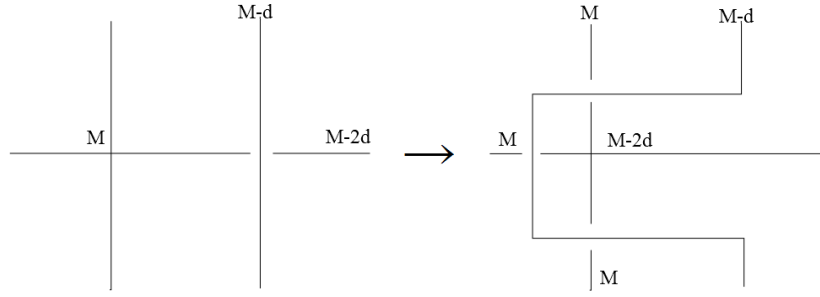


FIGURE 4.

Next, we transform the diagram inductively near the crossings colored by  $\{M|M-d|M-2d\}$  as shown in Figure 5.

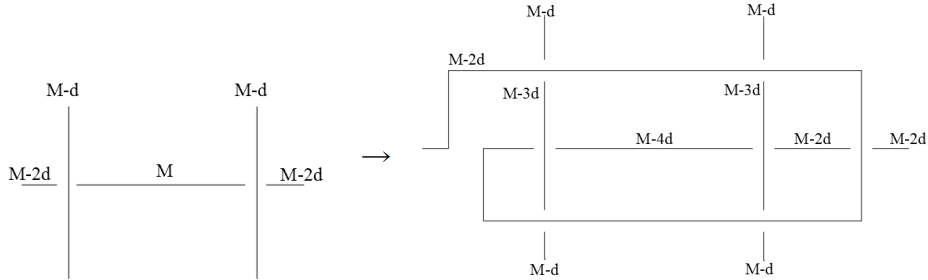


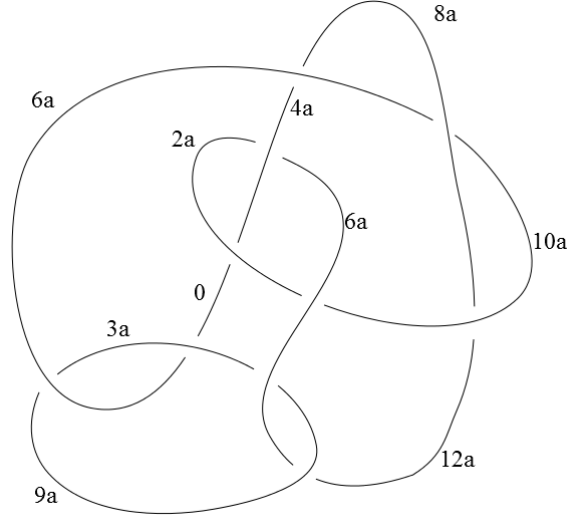
FIGURE 5.

Then we obtain a simple  $\mathbb{Z}$ -coloring such that the maximum of colors is less than that for the previous one.

Note that after changing the diagram, the common difference is still  $d$  or  $0$ . If the maximum of colors for the obtained  $\mathbb{Z}$ -coloring is at least  $4d$ , then we perform the transformation repeatedly. After all, we obtain a  $\mathbb{Z}$ -coloring with  $\{0, d, 2d, 3d\}$ . Therefore  $mincol_{\mathbb{Z}}(L)$  is at most 4. From Lemma 2.2, we see  $mincol_{\mathbb{Z}}(L) = 4$ .  $\square$

5.  $\mathbb{Z}$ -COLORING BY FIVE COLORS

In the previous section, we investigated the case that a diagram with a simple  $\mathbb{Z}$ -coloring. However, there are many diagrams of  $\mathbb{Z}$ -colorable links without simple  $\mathbb{Z}$ -colorings. See Figure 6 for example. In this section, we focus on the case that a diagram is colored by five colors. We first see the following in this case.


 FIGURE 6. Non-simple  $\mathbb{Z}$ -coloring for  $L10n32$ 

**Theorem 5.1.** *If a non-splittable  $\mathbb{Z}$ -colorable link admits a  $\mathbb{Z}$ -coloring with five colors, then there exists a  $\mathbb{Z}$ -coloring  $\gamma$  for  $L$  with  $\text{Im}(\gamma) = \{0, 1, 2, 3, 4\}, \{0, 1, 2, 3, 5\}, \{0, 1, 2, 3, 6\}, \{0, 1, 2, 4, 7\}, \{0, 2, 3, 4, 5\}, \{0, 3, 4, 5, 6\}$  or  $\{0, 3, 5, 6, 7\}$ .*

To prove the theorem above, we use the following notion, which is originally introduced for Fox coloring in [11].

**Definition 5.2.** For a  $\mathbb{Z}$ -coloring  $\gamma$  of a diagram of a  $\mathbb{Z}$ -colorable link, let  $c_1, c_2, \dots, c_l$  be the distinct colors on the arcs  $x_1, x_2, \dots, x_n$ . We define the *palette graph*  $G$  associated with  $\gamma$  as follows;

- (1) The vertices of  $G$  are the colors  $c_1, c_2, \dots, c_l$ .
- (2) Two vertices  $c_i$  and  $c_j$  are connected by an edge if and only if there is a crossing of the diagram whose under arcs are colored by  $c_i$  and  $c_j$ . We label the edge in (2) by  $(c_i + c_j)/2$ , which is coincident with one of  $c_1, c_2, \dots, c_l$  assigned to the over arc.

For a  $\mathbb{Z}$ -coloring on a diagram of a link with at least 2 components, the palette graph can be disconnected. For example, Figure 7 illustrate the palette graph for the link  $L10n36$  with a  $\mathbb{Z}$ -coloring.

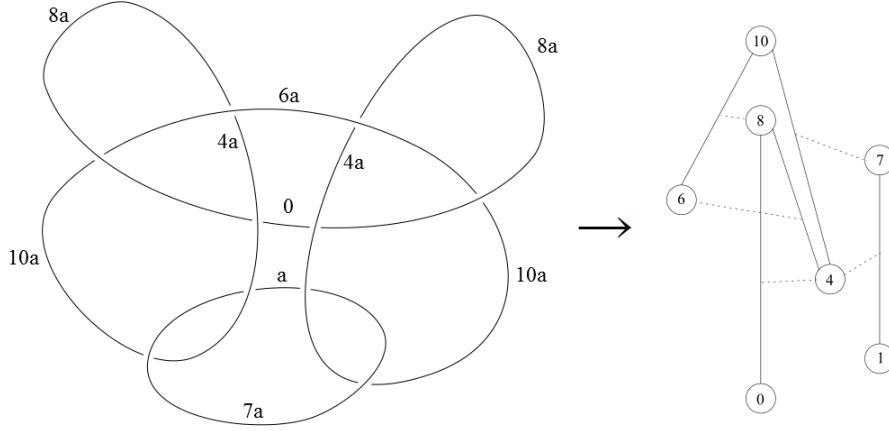


FIGURE 7. Palette graph for  $L10n36$

*Proof of Theorem 5.1.* We suppose that a non-splittable  $\mathbb{Z}$ -colorable link admits a  $\mathbb{Z}$ -coloring  $\gamma$  with five colors. By Lemma 2.2, without loss of generality, we assume that the image of  $\gamma$  is  $\{0, a, b, c, d\}$  with  $0 \neq a \neq b \neq c \neq d$ . There are even numbers and odd numbers in the image of  $\gamma$ . A vertex labeled by an even number and a vertex labeled by an odd number are not connected by definition of the palette graph. Since the link is non-splittable, there are at least 2 vertices with even numbers and at least 2 vertices with odd numbers. Furthermore any vertex is connected to another vertex. Therefore the palette graph of the diagram is a 2-component graph.

In the case that the vertices colored by  $0, a$  are connected and the vertices colored by  $b, c, d$  are connected as shown in Figure 8, we obtain the cases that  $Im(\gamma) = \{0, 1, 2, 3, 5\}$  and  $\{0, 3, 5, 6, 7\}$ .



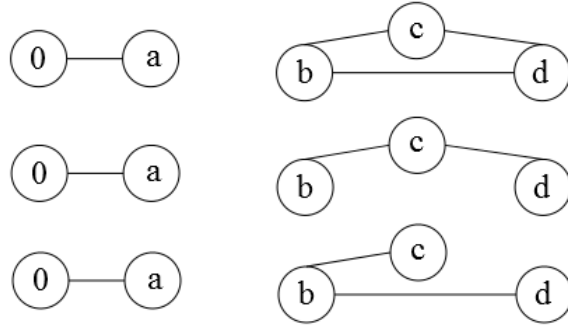


FIGURE 8.

In the case the vertices colored by  $0, a, b$  are connected and the vertices colored by  $c, d$  are connected as shown in Figure 9, we obtain the cases that  $Im(\gamma) = \{0, 1, 2, 3, 4\}$ ,  $\{0, 1, 2, 3, 6\}$ ,  $\{0, 2, 3, 4, 5\}$ ,  $\{0, 3, 4, 5, 6\}$  and  $\{0, 1, 2, 4, 7\}$ .

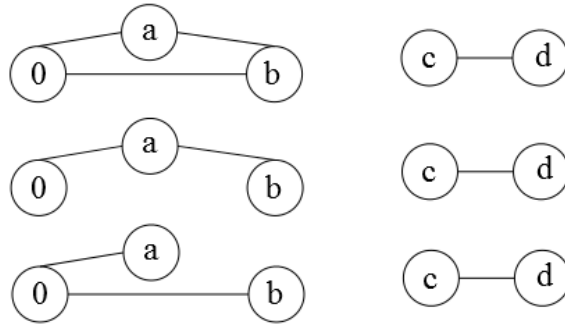
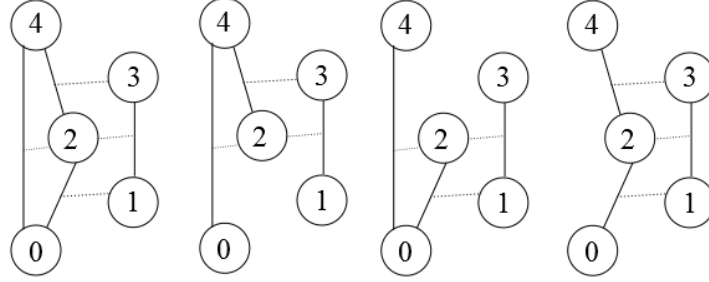


FIGURE 9.

□

**Remark 5.3.** *There exist four palette graphs associated with a  $\mathbb{Z}$ -coloring  $\gamma$  with  $Im(\gamma) = \{0, 1, 2, 3, 4\}$  as shown in Figure 10.*

FIGURE 10. Palette graphs with the image  $\{0, 1, 2, 3, 4\}$ 

The following is our second main theorem.

**Theorem 5.4.** *If a non-splittable link  $L$  admits a  $\mathbb{Z}$ -coloring  $\gamma$  with five colors, then  $\text{mincol}_{\mathbb{Z}}(L) = 4$ .*

**Remark 5.5.** *In the case that  $\text{Im}(\gamma) = \{0, 1, 2, 3, 4\}$ , it cannot be assumed that  $\gamma$  is a simple  $\mathbb{Z}$ -coloring, as seen in Remark 5.3.*

*Proof of Theorem 5.4.* Let  $D$  be a diagram of a non-splittable link  $L$  and  $\gamma$  a  $\mathbb{Z}$ -coloring on  $D$ .

We first consider the case that  $\text{Im}(\gamma) = \{0, 1, 2, 3, 4\}$ .

We start with deleting the crossings colored by  $\{4|4|4\}$  as illustrated in Figure 11. In the figure,  $3(2)$  and  $2(0)$  indicate two cases of the colors of the corresponding arcs, that is, when one is 3 (resp. 2), then the other is 2 (resp. 0).

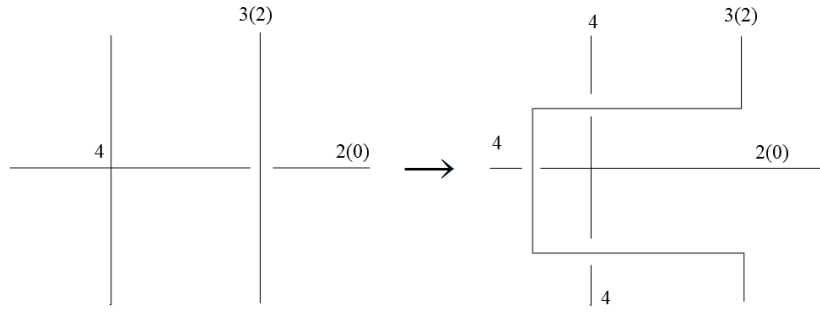


FIGURE 11.

Now, since 4 is the maximum in  $\text{Im}(\gamma)$ , the arc colored by 4 must be an under arc at any crossing. Then we will delete the crossings colored by  $\{4|*|*\}$  as follows.

In the case that the diagram has colored crossings shown in Figure 12, we transform the diagram and the coloring as shown in Figure 13 or 14.

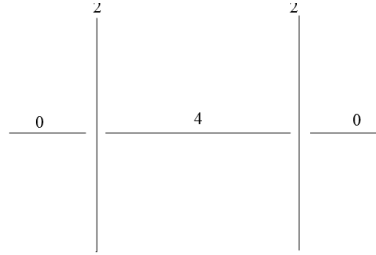


FIGURE 12.

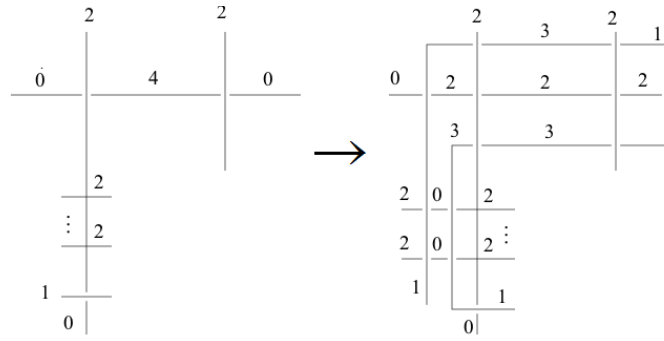


FIGURE 13.

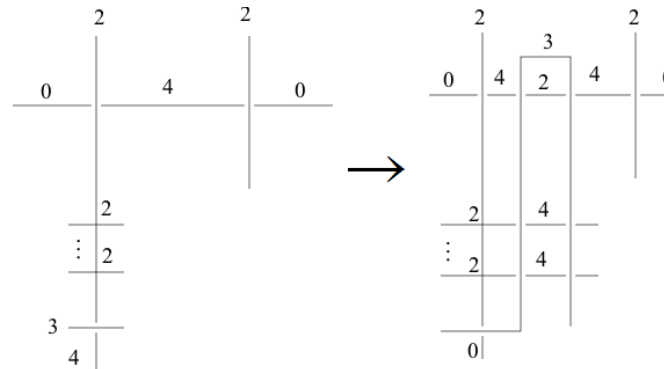


FIGURE 14.

In the case that a crossing colored by  $\{4|3|2\}$  exists, we transform the diagram and the coloring as illustrated in Figure 15 and 16.

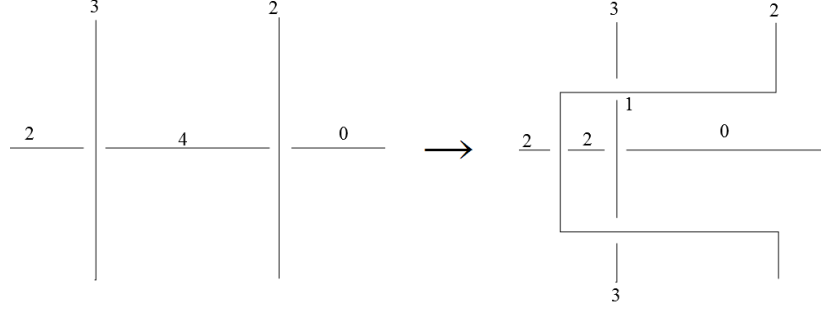


FIGURE 15.

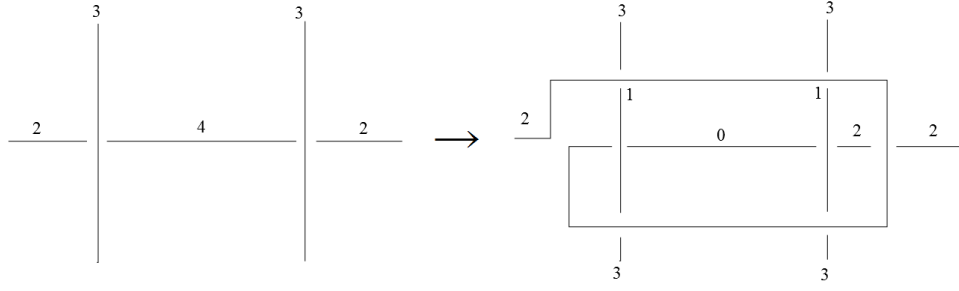


FIGURE 16.

Consequently we obtain a simple  $\mathbb{Z}$ -coloring. From Theorem 4.2, we see that  $\text{mincol}_{\mathbb{Z}}(L) = 4$ .

In the case that  $\text{Im}(\gamma) = \{0, 1, 2, 3, 5\}$ , first we delete crossings colored by  $\{5|5|5\}$  as shown in Figure 17.

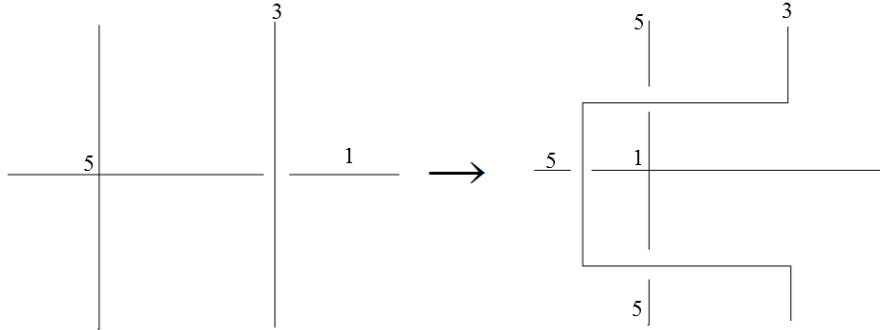


FIGURE 17.

Next we focus on a crossing colored by  $\{1|3|5\}$ . On the extension of the arc colored by 3, a crossing colored by  $\{3|2|1\}$  exists. Then we transform the diagram as shown in Figure 18.

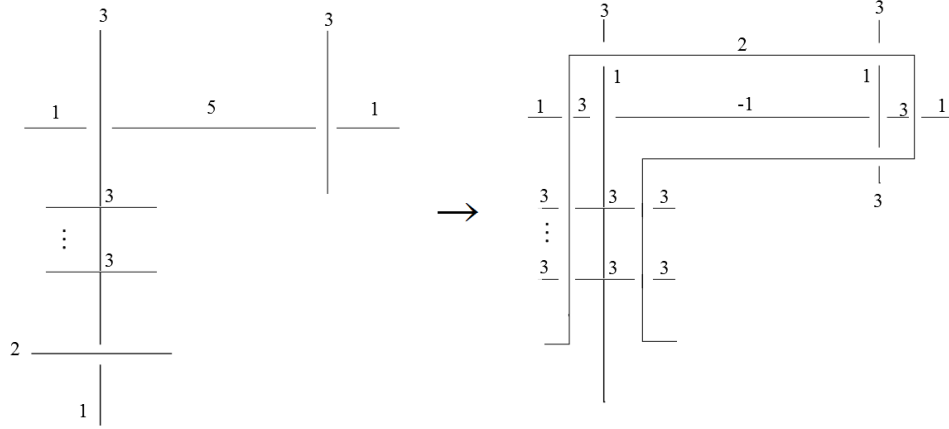


FIGURE 18.

Then we see that the diagram is colored by  $\{-1, 0, 1, 2, 3\}$ . We add 1 to all the colors, and then we obtain the colors  $\{0, 1, 2, 3, 4\}$ . Then we reduce this case to the case of  $Im(\gamma) = \{0, 1, 2, 3, 4\}$ , and we see  $mincol_{\mathbb{Z}}(L) = 4$ .

In the case that  $Im(\gamma) = \{0, 2, 3, 4, 5\}$ , we add  $-5$  and multiply  $-1$  to all colors for  $\gamma$ . Then this case is reduced to the case of  $Im(\gamma) = \{0, 1, 2, 3, 5\}$ .

In the case that  $Im(\gamma) = \{0, 3, 4, 5, 6\}$ , the palette graph associated with  $\gamma$  is shown in Figure 19.

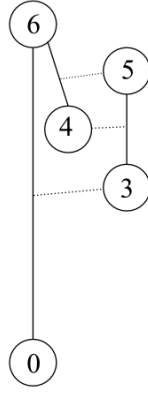


FIGURE 19.

We focus on an arc colored by 0. First we delete a crossing colored by  $\{0|0|0\}$  as shown in Figure 20.

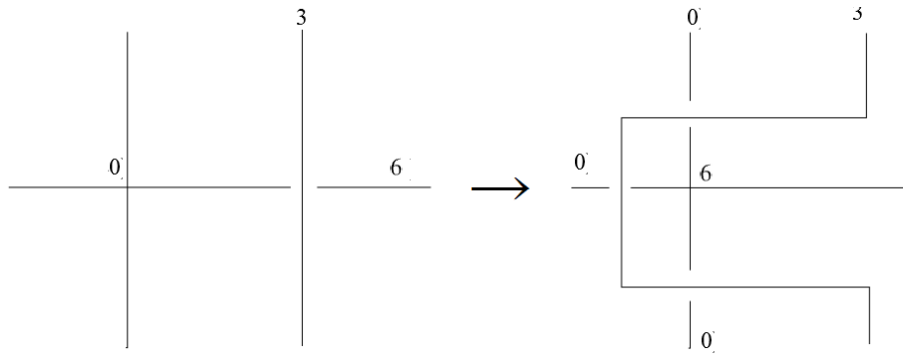


FIGURE 20.

Next we delete a crossing colored by  $\{0|3|6\}$  as shown in Figure 21.

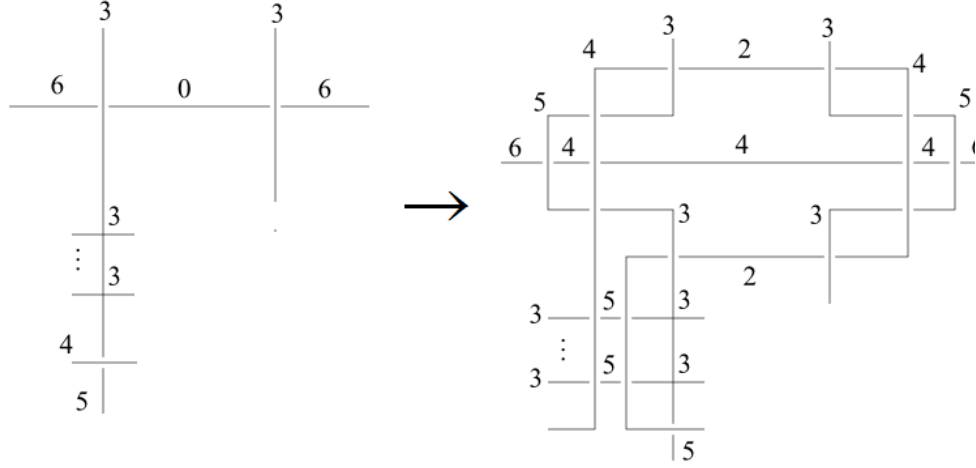


FIGURE 21.

Then we obtain another  $\mathbb{Z}$ -coloring  $\gamma'$  such that  $Im(\gamma') = \{2, 3, 4, 5, 6\}$ . We add  $-2$  to all the colors for  $\gamma'$ . Then we regard that this case is equivalence to the case  $Im(\gamma) = \{0, 1, 2, 3, 4\}$ .

In the case that  $Im(\gamma) = \{0, 1, 2, 3, 6\}$ , we add  $-6$  and multiply  $-1$  to all colors for  $\gamma$ . Then this case is reduced to the case of  $Im(\gamma) = \{0, 3, 4, 5, 6\}$ .

In the case that  $Im(\gamma) = \{0, 1, 2, 4, 7\}$ , the palette graph associated with  $\gamma$  is shown in Figure 22.

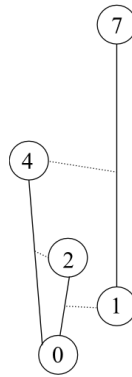


FIGURE 22.

First we delete a crossing colored by  $\{0|0|0\}$ . We transform the diagram and the coloring depicted in Figure 23. In the figure,  $1(2)$  and

$2(4)$  indicate two cases of the colors of the corresponding arcs, that is, when one is 1 (resp. 2), then the other is 2 (resp. 4).

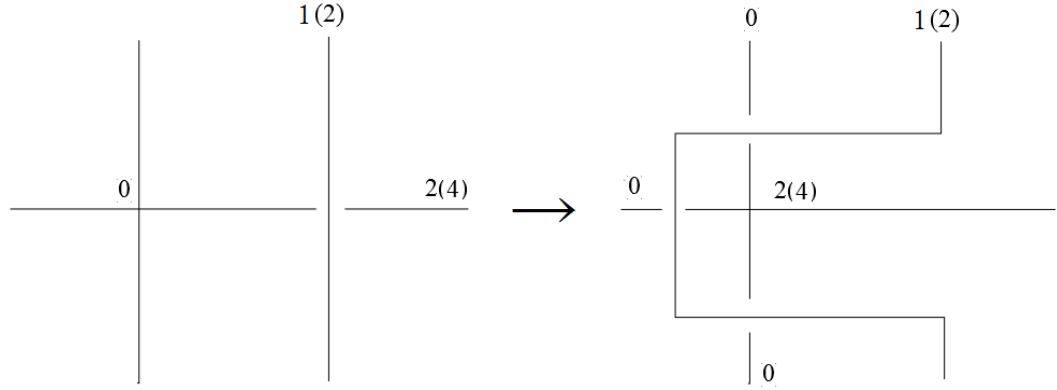


FIGURE 23.

Next we delete a crossing colored by  $\{0|2|4\}$ . We transform the diagram and the coloring as follows.

[1] In the case that, on the extension of the arc colored by 0, a crossing colored by  $\{0|2|4\}$  exists, we transform the diagram as shown in Figure 24.

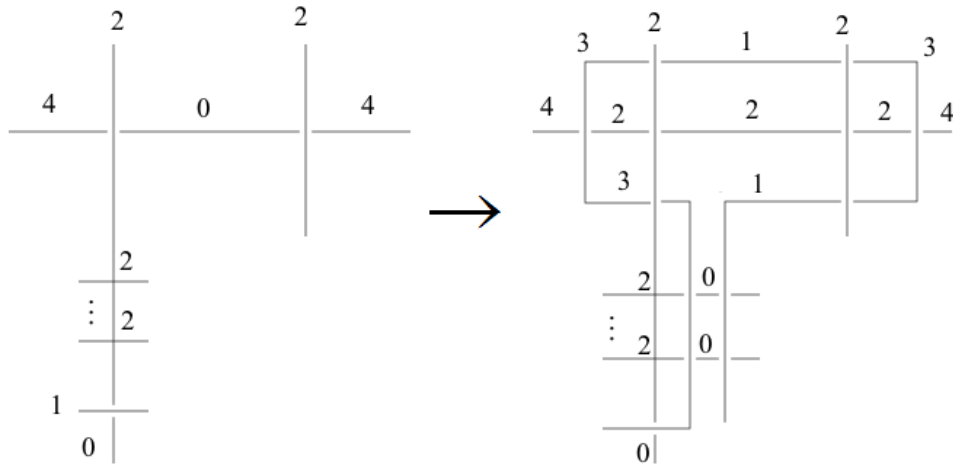


FIGURE 24.



[2] In the case that, on the extension of the arc colored by 0, a crossing colored by  $\{0|1|2\}$  exists, then we transform the diagram as shown in Figure 25.

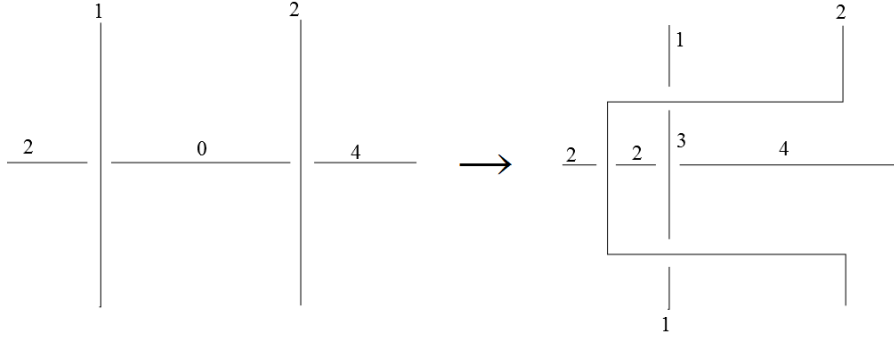


FIGURE 25.

After transformations [1] and [2], we obtain a palette graph as shown in Figure 26.

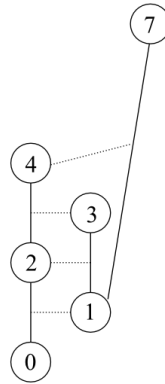


FIGURE 26.

[3] We now delete a crossing colored by  $\{7|4|1\}$ . On the extension of the arc colored by 4, a crossing colored by  $\{4|3|2\}$  exists. Then we transform the diagram as shown in Figure 27.

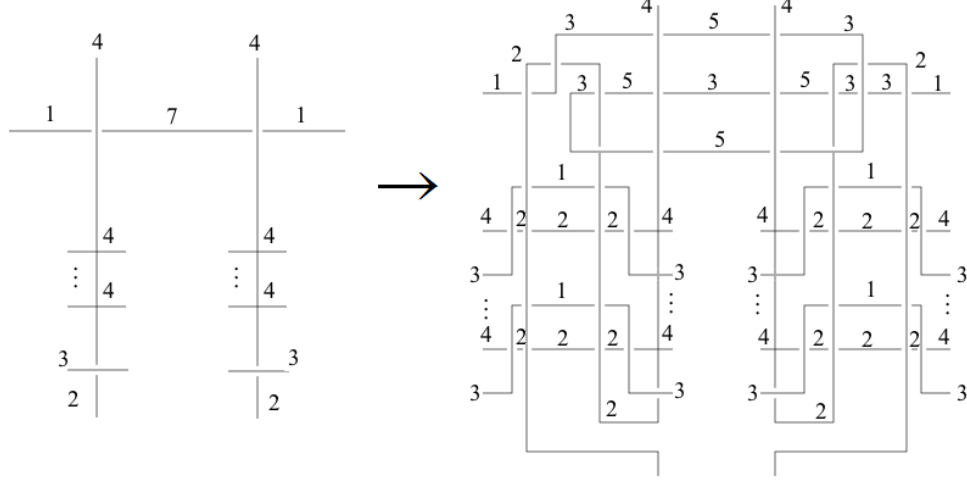


FIGURE 27.

Then we obtain another simple  $\mathbb{Z}$ -coloring  $\gamma'$  such that  $Im(\gamma') = \{0, 1, 2, 3, 4, 5\}$ . From Theorem 4.2, we see  $mincol_{\mathbb{Z}}(L) = 4$ .

In the case that  $Im(\gamma) = \{0, 3, 5, 6, 7\}$ , we add  $-7$  and multiply  $-1$  to all colors for  $\gamma$ . Then this case is reduced to the case of  $Im(\gamma) = \{0, 1, 2, 4, 7\}$ .

Consequently we have completed our proof of Theorem 5.4.  $\square$

## 6. LINKS WITH SIMPLE $\mathbb{Z}$ -COLORINGS

In this section, we present diagrams of the links with at most 10 crossings with simple  $\mathbb{Z}$ -colorings. We first use the diagrams given in [1]. For each link, if the diagram admits only a non-simple  $\mathbb{Z}$ -coloring, then we modify it to admit a simple one.

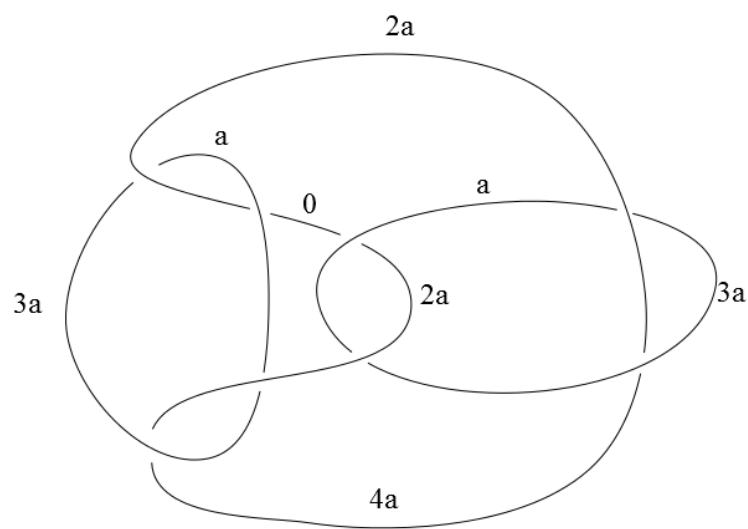


FIGURE 28.  $L8n6$

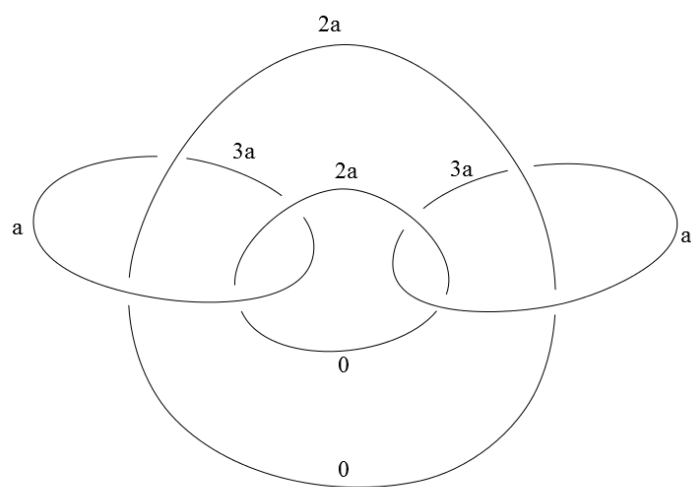
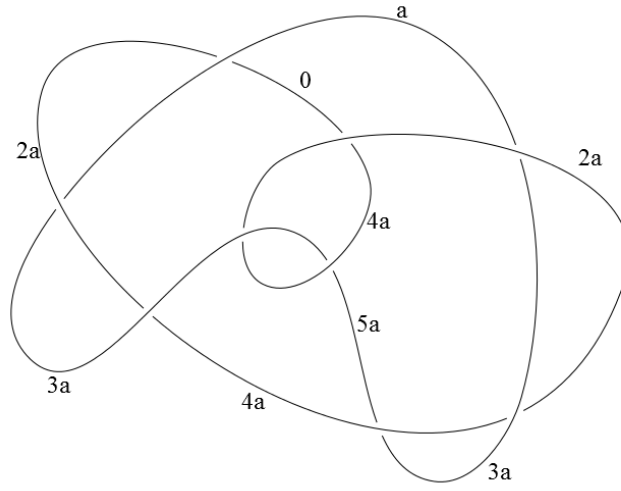
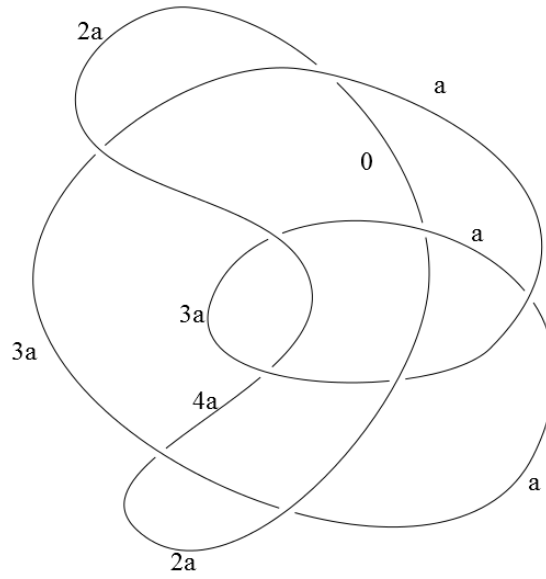
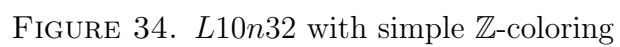
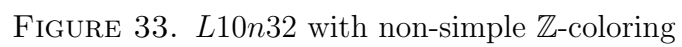
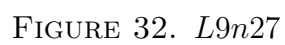
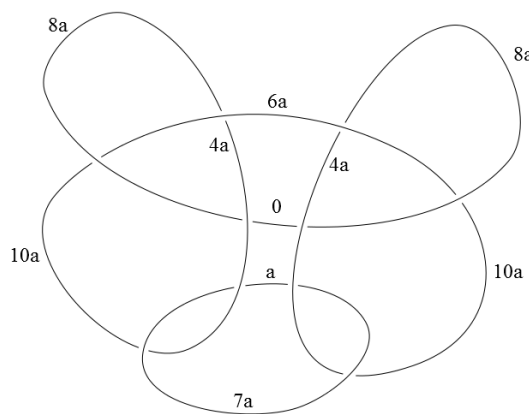
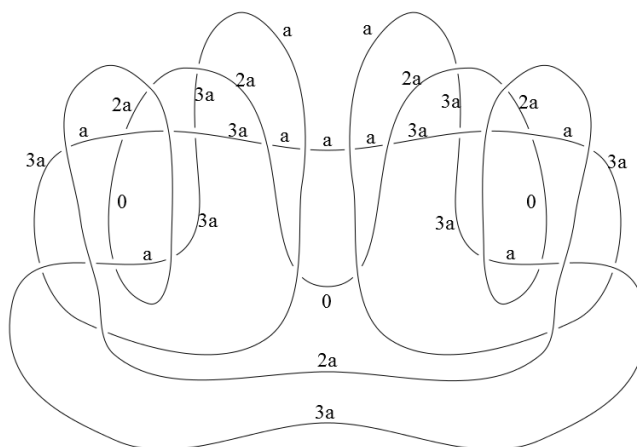
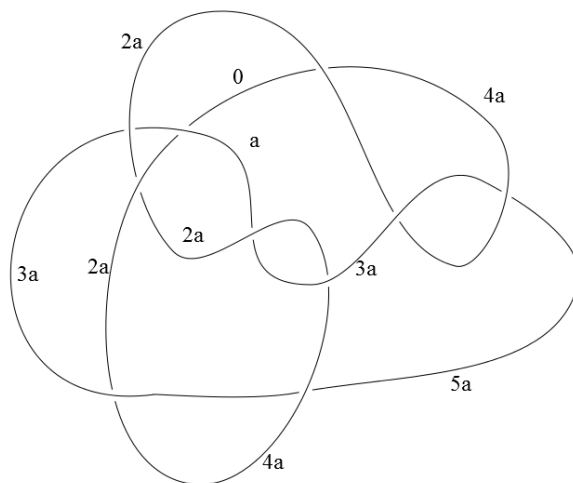
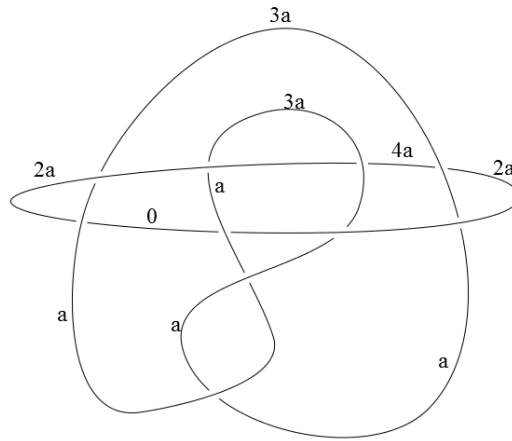
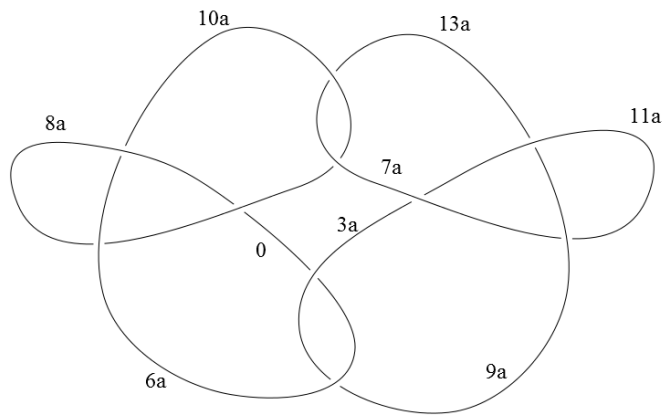
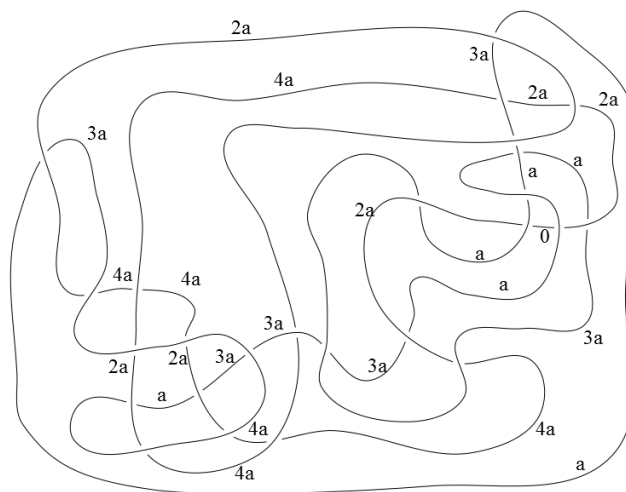


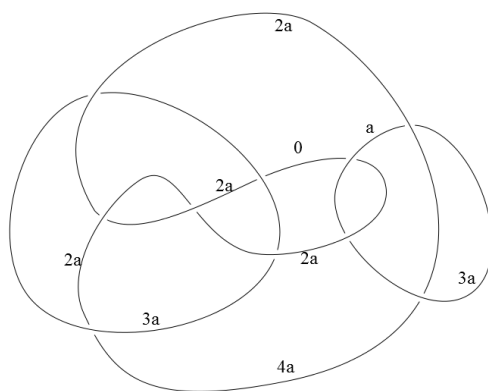
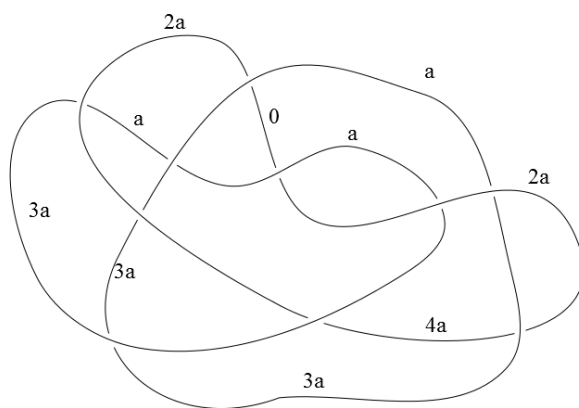
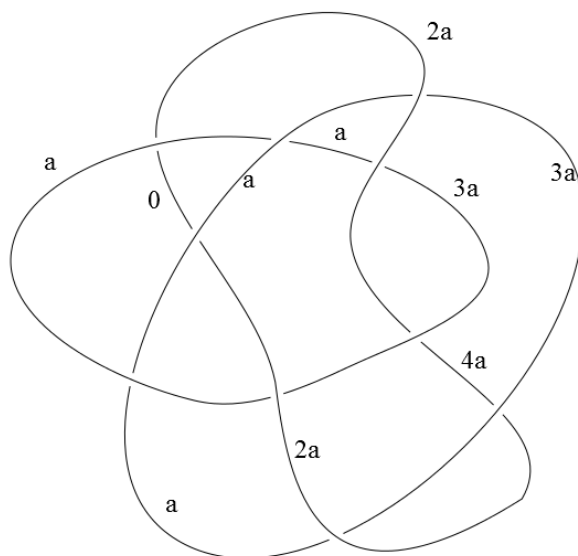
FIGURE 29.  $L8n8$

FIGURE 30.  $L9n18$ FIGURE 31.  $L9n19$

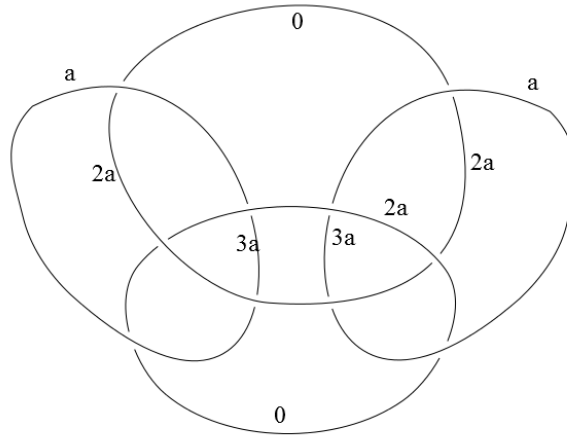
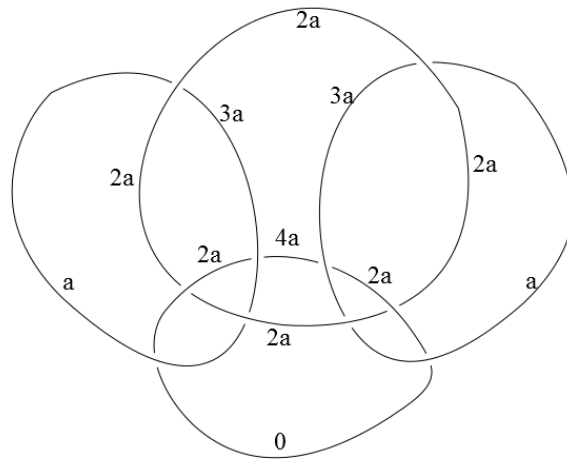
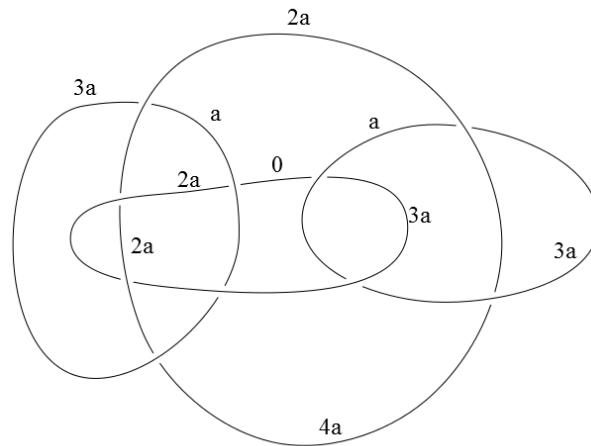


FIGURE 35.  $L10n36$ FIGURE 36.  $L10n36$  with simple  $\mathbb{Z}$ -coloringFIGURE 37.  $L10n56$


 FIGURE 38.  $L_{10n57}$ 

 FIGURE 39.  $L_{10n59}$  with non-simple  $\mathbb{Z}$ -coloring

 FIGURE 40.  $L_{10n59}$  with simple  $\mathbb{Z}$ -coloring

FIGURE 41.  $L_{10n91}$ FIGURE 42.  $L_{10n93}$ FIGURE 43.  $L_{10n94}$




 FIGURE 44.  $L10n104$ 

 FIGURE 45.  $L10n107$ 

 FIGURE 46.  $L10n111$

## 7. QUESTION

In view of our studies in the previous sections, it is natural to ask;

**Question 7.1.** *Does  $\text{mincol}_{\mathbb{Z}}(L) = 4$  always hold for any non-splittable  $\mathbb{Z}$ -colorable link  $L$ ?*

This is equivalent to the next by virtue of Theorem 4.2.

**Question 7.2.** *Does every non-splittable  $\mathbb{Z}$ -colorable link admit a simple  $\mathbb{Z}$ -coloring?*

## ACKNOWLEDGEMENT

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DEPARTMENT OF MATHEMATICS, COLLEGE OF HUMANITIES AND SCIENCES,  
NIHON UNIVERSITY, 3-25-40 SAKURAJOSUI, SETAGAYA-KU, TOKYO 156-8550,  
JAPAN

*E-mail address:* `ichihara@math.chs.nihon-u.ac.jp`

GRADUATE SCHOOL OF INTEGRATED BASIC SCIENCES, NIHON UNIVERSITY,  
3-25-40 SAKURAJOSUI, SETAGAYA-KU, TOKYO 156-8550, JAPAN

*E-mail address:* `cher16001@g.nihon-u.ac.jp`